Short-wavelength instability in systems with slow long-wavelength dynamics

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One-dimensional systems undergoing short-wavelength instability of spatially uniform states are studied. It is assumed that the spectrum of perturbations of the uniform states γ_k has a long-wavelength slowly-relaxing branch, detaching from a neutrally stable (Goldstone) mode with zero wave number, whose existence is a consequence of the problem's symmetry. The other important feature of the problem is quadratic nonlinearity that provides coupling between slowly-varying short-wavelength and long-wavelength modes. It is shown that the case is characterized by mixing of different scales in perturbative calculations. The latter makes the pattern stability problem essentially nonlocal and sensitive to very subtle characteristics of the spectrum γ_k and nonlinear mode-coupling. The equation governing longitudinal seismic waves in viscoelastic media is studied in detail as the simplest particular example of such systems. Possible extension of the obtained results to other physical problems, including electroconvection in a homeotropically aligned nematic layer and permeation of cholesterics or smectics in capillaries, is discussed. [S1063-651X(96)11910-3]

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INTRODUCTION

In the present paper the pattern formation problem in systems with short-wavelength instability and slow long-wavelength dynamics associated with the problem's symmetry is studied. One of the simplest realizations of the problem is connected with the equation

$$\frac{\partial v}{\partial t}+\frac{\partial^2}{\partial x^2}\left[\varepsilon-\left(1+\frac{\partial^2}{\partial x^2}\right)^2\right]v+v\,\frac{\partial v}{\partial x}=0. \tag{1}$$

The equation was proposed in Ref. [1] to describe propagation of longitudinal seismic waves in viscoelastic media. Here the real control parameter ε is supposed to be small and v is a scalar quantity, which has the meaning of the dimensionless displacement velocity.

The trivial solution of Eq. (1), $v\equiv 0$, being stable at $\varepsilon<0$, undergoes at $\varepsilon>0$ short-wavelength instability with respect to infinitesimal spatially periodic perturbations of the form

$$\delta v \propto \exp(\gamma_k t+ikx), \tag{2}$$

with wave numbers from a narrow band, centered around the point $k=1$.

Other examples of systems of such a kind may exhibit Rayleigh-Bénard convection with the so-called ''free-slip'' boundary conditions [2–4], systems with Galilean invariance [5], traveling front in phase transition phenomena or in reaction-diffusion systems [6,7], and others [8].

Among the variety of problems there are two of special interest. Both of them belong to physics of liquid crystals. The first problem corresponds to electroconvection in a nematic layer with homeotropic boundary conditions [9–12], the second to the so-called *permeation* of cholesterics or smectics at their motion through capillaries [13,14]. We will return to these two problems in Sec. IV of the present paper.

The main advantage of Eq. (1) compared to these examples is its relative simplicity. The latter provides the opportunity to see quite clearly the basic features of the phenomenon that are not obscured by a number of minor details, and to trace back close connections of these features with the problem's symmetry.

Equation (1) was already studied by Malomed [15]. Starting from this equation, he arrived at the system of the coupled generalized Ginzburg-Landau equations for slowly-varying amplitudes and analyzed solutions of the system and their stability. However, the generalized Ginzburg-Landau equations considered in Ref. [15] are inadequate to study underlying Eq. (1). The point is that Eq. (1) has additional (compared to the conventional spatiotemporal translations and spatial reflections) symmetry. The symmetry gives rise to certain peculiarities of perturbative calculations on this equation of the same nature as those discussed earlier in the case of the free-slip convection [3,4]. As a result, some corrections to the amplitude equations higher-order in ε, omitted in Ref. [15], contribute terms of leading order to the final dispersion equation in the pattern stability problem (ε*-scale-mixing*). In what follows the analysis of Eq. (1) free from the above-mentioned incorrectness of Ref. [15] is developed. It is shown that at small ε all steady spatially periodic solutions of Eq. (1) are unstable, contrary to Ref. [15], where the finite range of stability was found.

However, the main goal of the present paper is not to correct the results of Ref. [15]. The goal is, considering Eq. (1) as the simplest particular example of the systems with short-wavelength instability and additional continuous group of symmetry, to call attention to the fact that properties of

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these systems differ from conventional so drastically that it is worth singling them out into a separate class of pattern-forming systems.

The structure of the paper is as follows. In Sec. I a family of steady specially periodic solutions to equations of the type of Eq. (1) is obtained. In Sec. II the peculiarities of the stability analysis of these solutions related to the ε-scale-mixing are discussed and the origin of the mixing is revealed. In this section the adequate approach to the stability problem is developed. The section ends in the derivation of the dispersion equation for small perturbations of the steady spatially periodic solutions. In Sec. III the dispersion equation is analyzed. Section IV is devoted to general discussion of the results.

I. SPATIALLY PERIODIC PATTERNS

First of all let us show that Eq. (1) does possess the additional symmetry. With this end in view we consider the equation

$$\frac{\partial u}{\partial t}+\frac{\partial^2}{\partial x^2}\left[\varepsilon-\left(1+\frac{\partial^2}{\partial x^2}\right)^2\right]u+\left(\frac{\partial u}{\partial x}\right)^2=0. \tag{3}$$

Differentiating it with respect to x and renaming $2u_x$ as v, we reduce Eq. (3) to Eq. (1); that is to say, both the equations are equivalent. Note now that besides the conventional symmetry transformations

$$t\to t+\text{const}, \tag{4}$$

$$x\to x+\text{const}, \tag{5}$$

$$x\to -x, \tag{6}$$

Eq. (3) is invariant under the transformation

$$u\to u+\text{const}. \tag{7}$$

Explicit manifestation of this additional symmetry makes Eq. (3) much more convenient for the subsequent analysis compared to Eq. (1).

Linearizing Eq. (3) about the trivial state $u\equiv 0$ and taking the perturbations in the same form as that in Eq. (2), we can easily obtain the spectrum γ_k. The spectrum is

$$\gamma_k=k^2[\varepsilon-(k^2-1)^2]. \tag{8}$$

Qualitative difference between expression (8) and the corresponding spectra of the conventional problems, such as Rayleigh-Bénard convection with the rigid (no slip) boundary conditions [8], is vanishing of γ_k at $k=0$. In turn, the vanishing is an apparent consequence of symmetry transformation (7) that generates a neutrally stable (*Goldstone*) mode $\delta u=\text{const}$ in the spectrum of perturbations of the trivial state.

At $\varepsilon>0$ there is a band of unstable perturbations, whose boundaries $k_{1,2}$ are defined by the conditions $\gamma_{k_{1,2}}=0, \gamma_k>0$ at $k_1<k<k_2$. In particular, at $0<\varepsilon\ll 1$ we obtain

$$k_{1,2}-1=\mp\frac{\sqrt{\varepsilon}}{2}[1+O(\sqrt{\varepsilon})]. \tag{9}$$

Looking for nonlinear spatially periodic solutions to Eq. (3), it is natural to represent them in the form of Fourier series, i.e.,

$$u(x,t)=\sum_{n=-\infty}^{\infty}U_{nk}(t)e^{inkx},\quad U_{nk}^{*}=U_{-nk}. \tag{10}$$

Substituting Eq. (10) into Eq. (3), we arrive at the following set of coupled equations for the Fourier coefficients (*amplitudes*) U_{nk} with $n\neq 0$:

$$\frac{dU_{nk}}{dt}=\gamma_{nk}U_{nk}+k^2\sum_{l=-\infty}^{\infty}l(n-l)U_{lk}U_{(n-l)k}, \tag{11}$$

and a detached equation for $U_0(t)$ obtained from this set at $n=0$:

$$\frac{dU_0}{dt}=-k^2\sum_{l}l^2|U_{lk}|^2. \tag{12}$$

Thus, the amplitude $U_0(t)$ is slaved to those with $n\neq 0$. An interesting consequence of Eq. (12) is that any nontrivial dynamics yields *negative* corrections to $U_0(t)$. However, being x-independent, the mode with $n=0$ plays no role in pattern formation and always may be excluded from the problem by means of the transformation $\tilde{u}(x,t)=u(x,t)-U_0(t)$.

As for the amplitudes with $n\neq 0$, truncating the set of equations at finite $|n|>2$, it is easy to find that, at any k from the segment $k_1<k<k_2$, there is a steady solution to these equations of the form

$$U_{nk}=\sum_m U_{nk}^{(m)},\quad U_{nk}^{(m)}=O(\varepsilon^{(|n|+2m)/2});$$
$$n\neq 0,\quad m=0,1,2,\ldots. \tag{13}$$

For example, at $n=\pm 1,\pm 2$, we obtain

$$|U_k|^2=-\frac{\gamma_k\gamma_{2k}}{4k^4}[1+O(\varepsilon)], \tag{14}$$

$$U_{\pm 2k}=-\frac{k^2U_{\pm k}^2}{\gamma_{2k}}[1+O(\varepsilon)]. \tag{15}$$

Note that at small ε and $k_1<k<k_2$ the quantity γ_k, being positive, has the order ε, while $\gamma_{2k}<0$ and is of order 1.

The errors of Eqs. (14) and (15) are connected with the truncation procedure exclusively — expressions for γ_k and γ_{2k} are regarded here as *exact* ones, given by Eq. (8). On the other hand, since Eqs. (14) and (15) are valid only at $\varepsilon\ll 1$, the expressions for γ_k and γ_{2k} may be expanded in powers of small difference $\kappa\equiv k-1$, see Eq. (9). In this case to lowest order in the expansion parameter Eqs. (14) and (15) read as follows:

$$|U_k|^2=9(\varepsilon-4\kappa^2)[1+O(\sqrt{\varepsilon})], \tag{16}$$

$$U_{\pm 2k}=\frac{U_{\pm k}^2}{36}[1+O(\sqrt{\varepsilon})] \tag{17}$$

[note the different accuracy of Eqs. (14) and (15) and Eqs. (16) and (17)].

II. PATTERN STABILITY PROBLEM

As it was already mentioned in the Introduction, stability analysis of the steady spatially periodic solutions obtained in the preceding section requires high accuracy of intermediate calculations. To arrive at the right stability conditions in the case of the ε-scale-mixing we must consecutively increase accuracy of intermediate calculations until the stability conditions stop to change in a leading approximation. However, later we will see that only a few higher-order terms in intermediate expressions produce a contribution of leading order to the final stability conditions, while all other corrections are negligible. For this reason to simplify the calculations it is important to select among the variety of equivalent approaches to the pattern stability problem the most effective one, which allows us to detect the ''crucial'' corrections without calculations of those that may be neglected. In the generally accepted method of slowly-varying amplitudes each higher-order term has its *individual* structure and therefore has to be calculated *explicitly*. The latter makes rather inconvenient application of the method to the problem under consideration. Much more beneficial is representation of the steady solution in the form of Eqs. (10) and (13), where explicit expressions for the amplitudes U_{nk} are not used until the final stage of calculations, while perturbations of this solution are written as follows:

$$\delta u = e^{\sigma t} \sum_{n} V_{nk+p} e^{i(nk+p)x}, \tag{18}$$

where all the coefficients V_{nk+p} are constants.

Linearizing Eq. (3) about the unperturbed solution, we easily derive the following set of equations for V_{nk+p}:

$$(\sigma - \gamma_{nk+p}) V_{nk+p} - 2 \sum_{l} (n-l)(lk+p) k U_{(n-l)k} V_{lk+p} = 0. \tag{19}$$

Note that besides the above-mentioned convenience of calculations such an approach provides the opportunity to obtain the final stability conditions, employing just qualitative features of the spectral curve γ_k, without the concrete definition of its particular form that, actually, is a natural generalization of the problem.

Truncation of the set of Eq. (19) is the key point of the analysis. Therefore let us discuss it in detail. Formally the procedure is trivial: taking some integer N and dropping all V_{nk+p} with $|n|>N$, we obtain a system of $2N+1$ linear equations for remaining V_{nk+p}. Next, as usual, the dispersion equation for $\sigma(p,k)$ is obtained by equating the determinant of this system to zero. Then, N increases by one *with simultaneous increase of accuracy of calculations of amplitudes* $U_{(n-l)k}$, the routine is repeated, and so on. The question is to find the adequate value of N to terminate the routine.

As always, it is reasonable to expect for unstable perturbations that $\sigma = O(\varepsilon)$ and $p = O(\sqrt{\varepsilon})$ — a guess that will be verified later, see Sec. III. Besides, at $p = O(\sqrt{\varepsilon})$ and $k_1 < k < k_2$ [i.e., $\kappa = O(\sqrt{\varepsilon})$ or less than that, see Eq. (16)] the quantities γ_{nk+p} are of order ε at $n = 0, \pm 1$ and of order 1 $[\gamma_{nk+p} = -n^2(n^2-1)^2 + O(\sqrt{\varepsilon})]$ at any other values of n, see Eqs. (8) and (9). Thus, σ in Eq. (19) cannot be neglected compared to γ_{nk+p} at least at three values of n, namely, at $n = 0, \pm 1$, which correspond to the wave numbers p and $\pm k + p$, respectively, i.e., the lowest approximation to N to begin with is $N = 1$. It yields a *cubic* dispersion equation for σ. Taking into account the guess $\sigma = O(\varepsilon)$, we arrive at the conclusion that the discussed dispersion equation must be accurate *at least* to $O(\varepsilon^3)$ inclusively.

The dispersion equation at $N = 1$ reads as follows:

$$\begin{vmatrix} \sigma - \gamma_{-k+p} & 2kpU_{-k} & 4k(k+p)U_{-2k} \\ -2k(-k+p)U_k & \sigma - \gamma_p & 2k(k+p)U_{-k} \\ -4k(-k+p)U_{2k} & -2kpU_k & \sigma - \gamma_{k+p} \end{vmatrix} = 0. \tag{20}$$

According to the general routine, now we have to increase N considering, step by step, determinants 5×5, 7×7, etc., where each next is obtained from previous by ''framing'' it in one additional top and bottom row, and one left and right column. Note now that all elements of the ''frames'' except those on the leading diagonal have a certain smallness in ε. As for the elements on the leading diagonal, they have the form $\sigma - \gamma_{\pm nk+p}$, where $\sigma = O(\varepsilon)$ and is small compared to $\gamma_{\pm nk+p}$ [we recall $\gamma_{\pm nk+p} = O(1)$ at $|n|>1$]. It means that at any $N>1$ terms of lowest order (both in ε and in σ) generated by evaluation of the corresponding determinant, are products of $\gamma_{\pm nk+p}$ from the leading diagonal of the ''frames'' and determinant (20). In other words, increase of N does not change the lowest order in ε of the dispersion equation, compared to Eq. (20), i.e., the *necessary* accuracy $O(\varepsilon^3)$ simultaneously is *sufficient* to the lowest approximation.

Proceeding with practical calculations it is convenient to transform rows of the ''frame'' to those of a triangular matrix. In this case the dispersion equation at any $N>1$ is reduced to the form of Eq. (20), where the elements of the determinant receive certain corrections. Physically such a procedure corresponds to exclusion of slaved modes.

At the moment let us pay attention to a remarkable peculiarity of determinant (20): the lowest order of terms its evaluation yields is not ε^3 — it is $\varepsilon^{5/2}$, see, e.g., the product $[4k(k+p)U_{-2k}]2kpU_k[2k(-k+p)U_k]$. Note that even if terms of order $\varepsilon^{5/2}$ cancel each other out entirely, it may not be the case for *corrections* to these terms. The latter means that to arrive at the dispersion equation with accuracy $O(\varepsilon^3)$ we must take into account *all corrections* with relative smallness to order $\sqrt{\varepsilon}$ to all elements of determinant (20) but the one standing on the middle of the leading diagonal, i.e., $\sigma - \gamma_p$. In particular, the relative smallness of dropped terms in expressions (14) and (15) is $O(\varepsilon)$ and therefore these expressions may be used in their present form, while the accuracy of Eqs. (16) and (17) is not sufficient for our purposes. The discussed peculiarity of determinant (20) is the actual grounds for the ε-scale-mixing.

It is a matter of straightforward calculations to obtain that under the specified accuracy, reduction of the ''frame'' to a ''triangular'' form generates corrections only to the two marginal elements of the leading diagonal of determinant (20), and the corrections do not change at $N>2$. The corrected elements read as follows:

$$\sigma-\gamma_{\pm k+p}\to\sigma-\gamma_{\pm k+p}-\frac{4k^2(p\pm k)(p\pm 2k)|U_k|^2}{\gamma_{\pm 2k+p}}$$

This replacement in Eq. (20) yields the desirable dispersion equation.

Expanding $\gamma_{\pm k+p}$ in powers of small p and $\kappa\equiv k-1$ [we recall $p=O(\sqrt{\varepsilon})$ and κ is not greater than that], evaluating the determinant, and dropping terms of order higher than ε^3, after trivial but rather tedious calculations we arrive at the following dispersion equation:

$$\sigma^3+a_1\sigma^2+a_2\sigma+a_3=0, \tag{21a}$$

$$a_1(k,p)=2\gamma_k-\gamma_1''p^2-\gamma_p, \tag{21b}$$

$$a_2(k,p)=-[(2\gamma_2+\gamma_1'')\gamma_k+(\gamma_k')^2]p^2 -(2\gamma_k-\gamma_1''p^2)\gamma_p+\left(\frac{\gamma_1''}{2}\right)^2p^4, \tag{21c}$$

$$a_3(k,p)=-\frac{2}{k}\gamma_{2k}\gamma_k\gamma_k'p^2+4(\gamma_2-\gamma_2')\gamma_k^2p^2 +\left(\gamma_1''-\frac{\gamma_1'''}{3}\right)\gamma_2\gamma_kp^4 +\left\{[(\gamma_k')^2+\gamma_1''\gamma_k]p^2-\left(\frac{\gamma_1''}{2}\right)^2p^4\right\}\gamma_p. \tag{21d}$$

Here primes denote derivatives with respect to k and subscripts stand for the values of k, so that, for example, γ_1'' is the value of $d^2\gamma_k/dk^2$ at $k=1$ ($\kappa=0$), i.e., just a number of order 1; meanwhile $\gamma_p,\gamma_k,\gamma_k'$, and γ_{2k} (unspecified values of the subscripts) means that they are functions of p, k, and $2k$, respectively.

III. ANALYSIS OF THE DISPERSION EQUATION

First of all let us examine the dispersion equation (21) in the limit $p\to 0$ (*sideband* perturbations). In this limit Eq. (21) may be written in the following form:

$$\sigma^3+2\gamma_k\sigma^2+b_2p^2\sigma+b_3p^2=0, \tag{22}$$

where coefficients $b_{2,3}$ do not depend on p, being certain functions of k, whose explicit form may be easily obtained from comparison of Eq. (22) with Eqs. (21). The double root $\sigma_{1,2}=0$ of Eq. (22) at $p=0$ corresponds to two Goldstone modes originated in symmetry transformations (5) and (7), respectively. Building up the solution of Eq. (22) in powers of p, we can find the two *Goldstone branches*, detaching from these modes:

$$\sigma_{1,2}=\pm p\sqrt{-\frac{b_3}{2\gamma_k}}+\frac{p^2}{4\gamma_k}\left(\frac{b_3}{2\gamma_k}-b_2\right)+O(p^3). \tag{23}$$

Since γ_k is positive at $k_1<k<k_2$, stability of these branches are defined by sign of b_3, and by interplay of the coefficients $b_3/2\gamma_k$ and b_2 in the second term on the right-hand side of Eq. (23). The evident stability conditions are $2\gamma_kb_2>b_3>0$.

Let us focus attention on the coefficient b_3. It is easy to see from Eq. (21d) that

$$b_3=-\frac{2}{k}\gamma_{2k}\gamma_k\gamma_k'+4(\gamma_2-\gamma_2')\gamma_k^2.$$

Term $4(\gamma_2-\gamma_2')\gamma_k^2$ is obviously of order ε^2. As for term $-2\gamma_{2k}\gamma_k\gamma_k'/k$ it should be analyzed more carefully. We begin the analysis with the case $\kappa=O(\sqrt{\varepsilon})$. The leading approximation to k is 1 and to γ_{2k} it is $\gamma_2=O(1)$. The most general expression for γ_k may be written as $f(\kappa)(\varepsilon-c\kappa^2)$. Here $f(0)=1$ and $f'(0)$, c are constants of order 1, so that $\gamma_k=O(\varepsilon)$. For γ_k' we have $\gamma_k'=\gamma_1'+\gamma_1''\kappa+O(\kappa^2)$, where $\gamma_1'=O(\varepsilon)$ and $\gamma_1''=O(1)$, see above the expression for γ_k. At $\kappa=O(\sqrt{\varepsilon})$ it yields $\gamma_k'=\gamma_1''\kappa+O(\varepsilon)=O(\sqrt{\varepsilon})$. Finally, at the specified κ we obtain $-2\gamma_{2k}\gamma_k\gamma_k'/k=-2\gamma_2\gamma_1''\gamma_k\kappa+O(\varepsilon^2)=O(\varepsilon^{3/2})$, which generates to $b_3/2\gamma_k$ the following leading approximation: $b_3/2\gamma_k=-\gamma_2\gamma_1''\kappa+O(\varepsilon)=O(\sqrt{\varepsilon})$. Taking into account that, as it is clearly seen from Eq. (21c), $b_2=O(\varepsilon)$, we arrive at the conclusion that at $\kappa=O(\sqrt{\varepsilon})$ term b_2 on the right-hand side of Eq. (23) may be neglected compared to $b_3/2\gamma_k$. However, it immediately follows from Eq. (23) that the neglect of b_2 yields instability associated either with the first term on the right-hand side of this equation ($b_3/2\gamma_k<0$) or with the second one ($b_3/2\gamma_k>0$).

Approaching zero, κ becomes of order ε that makes the neglect of b_2 in Eq. (23) irrelevant. Thus, the case $\kappa=O(\varepsilon)$ should be considered separately. Note that due to the obtained instability of all the spatially periodic solutions at $\kappa=O(\sqrt{\varepsilon})$, the only opportunity for the solutions to be stable may be associated with the just specified values of κ of order ε. Since the result practically does not depend on the concrete expressions for $b_{2,3}$, it reflects a generic feature of the systems with additional symmetry — the stability band for spatially periodic solutions in these systems narrows dramatically from $O(\sqrt{\varepsilon})$ in the conventional cases [8] to $O(\varepsilon)$ [3,4,12,15,18].

Proceeding with the analysis at $\kappa=O(\varepsilon)$ it is more convenient to employ for γ_k the explicit expression (8), since general formulas (21b)–(21d) for the coefficients $a_{1,2,3}$ result in a very awkward dispersion equation (21a). Then, expanding $a_{1,2,3}$ in powers of κ, introducing new dimensionless variables

$$z\equiv\frac{\sigma}{\varepsilon},\quad \chi\equiv\frac{\kappa}{\varepsilon},\quad \zeta\equiv\frac{p}{\sqrt{\varepsilon}}, \tag{24}$$

and dropping terms higher order in ε that appear due to the decrease of κ from $O(\sqrt{\varepsilon})$ to $O(\varepsilon)$, we can reduce Eq. (21) to the following nondimensional form:

$$z^3+c_1z^2+c_2z+c_3=0, \tag{25a}$$

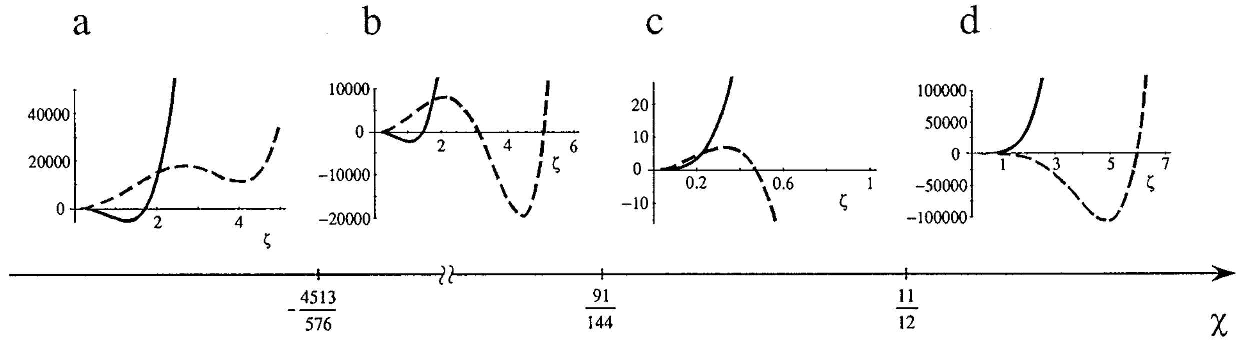

FIG. 1. Quantities T (—) and c_3 $(--)$ as functions of ζ at various fixed values of χ from different characteristic regions: (a) $\chi=-9<-4513/576$; (b) $-4513/576<\chi=-6<91/144$; (c) $91/144<\chi=0.7<11/12$ [second intersection of the curve $c_3(\zeta)$ with ζ axis lies far to the right from the point $\zeta=1$ and is not shown]; (d) $\chi=1>11/12$.

$$c_1\equiv 2+9\zeta^2,\quad c_2\equiv 2(41+12\zeta^2)\zeta^2,$$

$$c_3=48(11-12\chi)\zeta^2-568\zeta^4+16\zeta^6. \tag{25b}$$

Note that the fact that the dispersion equation may be reduced to a nondimensional form by the scale transformation (24) justifies the estimations of the characteristic values of σ and p made in the preceding section.

Comparison of Eqs. (22), (23), and (25) brings about the conclusion that the analyzed spatially periodic solutions of Eq. (3) are stable against the sideband perturbations if the quantity χ satisfies the conditions

$$\frac{91}{144}<\chi<\frac{11}{12}.$$

However, the stability against the sideband perturbations is only a *necessary* condition: it does not guarantee stability for disturbances with *finite* wave numbers. In other words instability may be connected with modes from a band separated from the Goldstone modes by a finite gap. For this reason the stability analysis should be extended to the case of arbitrary values of ζ. To do this it is convenient to employ the Routh-Hurwitz criterion [16]. Being applied to the problem under consideration, the criterion says the number of unstable branches of the spectrum described by Eq. (25) coincides with the number of changes of sign in the sequence

$$1, c_1, c_1 T, c_3, \tag{26}$$

where $c_{1,2,3}$ are defined according to Eq. (25b) and T stands for the following quantity:

$$T\equiv c_1c_2-c_3=200\zeta^6+1354\zeta^4+(576\chi-364)\zeta^2. \tag{27}$$

Since c_1 is strictly positive, the only possibility for sequence (26) to have negative terms may be associated with negativeness of T or/and c_3. It allows the following numbers of changes of sign in sequence (26): (i) zero, (ii) one, and (iii) two. Case (i) obviously corresponds to stable perturbations. Case (ii) means that Eq. (25) has *one* and *only one* root with positive real part. Since Eq. (25) has *real* coefficients, its complex roots must enter by a complex conjugate pair. One and only one root with positive real part means that this root must be *purely real*.

Generally speaking, two roots with positive real parts may mean either a pair of complex conjugate roots or two purely real roots. However, the left-hand side of Eq. (25) is a *monotonic* function of z at $z>0$ and hence this equation cannot have more than *one* purely real positive root. In other words case (iii) corresponds to a pair of *complex conjugate roots* with positive real part.

Note now that the quantity χ parametrizes a *given* spatially periodic solution, whose stability against *any* perturbation is studied. Therefore the most natural way to describe the stability spectrum is to present z as a function of ζ, considering χ as a parameter that may have different fixed value.

Trivial analysis of Eqs. (25b) and (27) shows that T has negative values inside a finite segment $0<\zeta<\zeta_T(\chi)$ at $\chi<91/144$, being positive at any $\zeta\neq 0$ when χ exceeds 91/144. As for the coefficient c_3, it is strictly positive at any $\zeta\neq 0$ if $\chi<-4513/576$. At $-4513/576<\chi<11/12$ there is a finite segment $\zeta_1(\chi)<\zeta<\zeta_2(\chi)$ where c_3 is negative. The left boundary of this segment $\zeta_1(\chi)$ is separated from the point $\zeta=0$ by a finite gap until χ remains smaller than 11/12. At $\chi=11/12$ the gap vanishes, so that at $\chi>11/12$ the segment of negative values of c_3 is defined by the inequalities $0<\zeta<\zeta_2(\chi)$ with finite $\zeta_2(\chi)$. The behavior of T and c_3 as functions of ζ at different values of χ is shown in Fig. 1.

Summarizing these results, we arrive at the following classification of the spectrum.

(i) $\chi<-4513/576$. Inside the domain $0<\zeta<\zeta_T(\chi)$ the signs in sequence (26) are $(++-+)$, which corresponds to two complex conjugate unstable Goldstone branches, i.e., to oscillatory instability.

(ii) $-4513/576<\chi<91/144$. The oscillatory instability of the same kind as before $(++-+)$, accompanied in the domain $\zeta_1(\chi)<\zeta<\zeta_2(\chi)$ by aperiodic instability $(+++-)$ with purely real growth rate. Note that $\zeta_1(\chi)$ is always greater than $\zeta_T(\chi)$, so that at any χ from the specified segment unstable oscillatory and aperiodic branches are separated from each other by a finite gap of stable perturbations.

(iii) $91/144<\chi<11/12$. The sign sequence is $(+++-)$, which means aperiodic instability in the domain

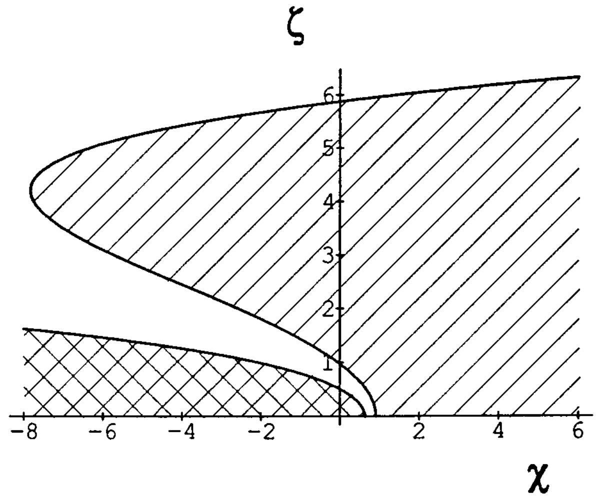

FIG. 2. The band structure of the spectrum of perturbations of steady spatially periodic solutions to Eq. (3) in the χ-ζ plane. Unhatched region corresponds to stable perturbations; hatched one indicates aperiodic instability; cross-hatched region shows the range of oscillatory instability. Note the gap between the two bands of instability at $\zeta=0$ and $91/144<\chi<11/12$.

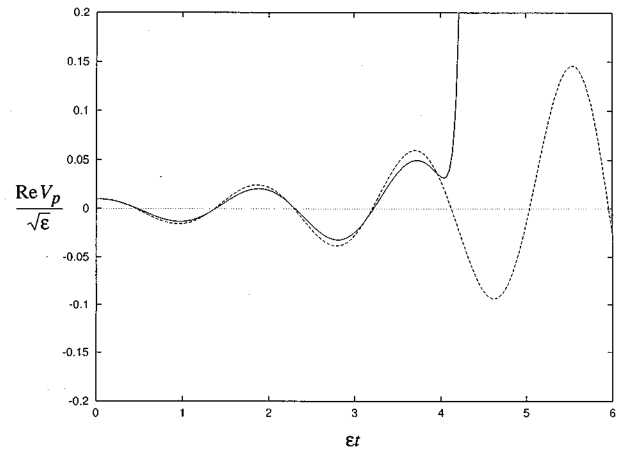

FIG. 3. Growth of an unstable eigenmode from the band of oscillatory instability of steady solution (13). The solid line corresponds to numerical integration of Eq. (3), the dashed one displays analytical expression $V_p(t)=V_p(0)\exp(\sigma t)$, where σ is given by solution of Eq. (25); $\varepsilon=10^{-4}$, $k=1$, $p=3.125\times10^{-3}$, $\mathrm{Re}V_p(0)/\sqrt{\varepsilon}=0.01$, $\mathrm{Im}V_p\equiv0$. The induction time is about $4/\varepsilon$ — note the sharp divergence of the curves at $\varepsilon t>4$ caused by nonlinear effects.

$\zeta_1(\chi)<\zeta<\zeta_2(\chi)$ — the only branch of unstable perturbations is separated from the Goldstone modes by the finite gap $0<\zeta<\zeta_1(\chi)$.

(iv) $\chi>11/12$. The same aperiodic branch $(+++-)$ detaches from one of the Goldstone modes $(\zeta_1=0)$.

Thus, we can see that at $91/144<\chi<11/12$ the system does possess a band of unstable modes which are not related to Goldstone branches. Since the solutions with χ, lying outside this segment, are unstable for the sideband perturbations, it means instability of all the spatially periodic solutions obtained in Sec. I. To complete the discussion of these properties of the spectrum, note that boundaries of different bands of instability are defined by the conditions $T=0$ and $c_3=0$, respectively, which yields the band structure shown in Fig. 2.

All results of the analytical study of the problem were checked against computer simulation, whose detailed description as well as discussion of the asymptotic state of the system at $t\to\infty$ were reported elsewhere [12,17,18]. In all cases for small perturbations the computed instability growth rates coincide quantitatively with the results of analytical consideration of the problem developed in the present paper (see Fig. 3 as an example) and contradict to those of Ref. [15].

It is also worth mentioning the results of computer simulations reported in Ref. [19]. The authors of this paper considered the equation of the type of Eq. (1) supplemented with the third spatial derivative of v, which adds to the dependence γ_k the imaginary part of the form $\mathrm{Im}\gamma_k\propto ik^3$. The dynamics described in Ref. [19] corresponded to transformation of white-noise-like initial conditions into spatially periodic patterns, which the authors identified as steady states. The results seem to be in contradiction with those discussed above. However, we have to emphasize that such a comparison of our results is irrelevant. Indeed, the third spatial derivative breaks left-right parity of the problem [symmetry transformation (6)], which may bring about drastic changes into the pattern stability spectrum [18]. Thus, any equation, whose operator includes odd spatial derivatives, requires a separate consideration, so the question about steady solutions to governing equation of Ref. [19] and their stability, actually, remains open.

Note besides that, strictly speaking, the numerical results of Ref. [19] are ambiguous and admit another interpretation. The authors of this work point out that their numeric code is stable only at $t<7/\mathrm{Re}\gamma_k$, i.e., the simulations cover just the very initial stage of pattern dynamics. On the other hand, all the patterns displayed in Ref. [19] as examples of the steady states exhibit quite clear long-wavelength modulations that may be regarded as the beginning of the instability discussed in the present paper. An additional argument in favor of this interpretation is that in *all* our simulations growth of unstable long-wavelength modes, initially rather slow, suddenly, after a certain induction time of several inverse ε, becomes very sharp (in the same time-scale, see Fig. 3) and gives rise to dramatic increase of the corresponding amplitudes. Remarkable agreement between the induction time and the stability limit of the code in Ref. [19] provides grounds to suppose that a similar change of the pattern dynamics may be the actual reason for the numeric instability in work [19].

Ending this section we would like to emphasize that since the discussed instability is connected with the growth of long-wavelength modes, the problem is very sensitive to cutoff of the spectrum caused by finiteness of spatial size of a real system (*size-effect*) that is important both for computer simulations and for experimental verification of the instability [12,17,18].

IV. GENERAL DISCUSSION

Naturally, the instability of all steady spatially periodic patterns governed by Eq. (3) is a specific peculiarity of this particular equation. However, the *fact* that a pattern-forming system may possess such a peculiarity at small values of the control parameter is a generic property of problems with additional symmetry [3,4,12,18] (we remind the reader that at small ε the conventional systems without slow long-wavelength dynamics always have a finite domain of stability of these patterns, see, e.g., [8]).

Another generic property of the problem is the ε-scale-mixing. As a result, dispersion equation (21) contains, in addition to the conventional quantity γ_1'', the derivatives γ_1''' and γ_2' [see Eq. (21d)], that enter into the leading approximation to the final stability conditions. Thus, the stability conditions depend on the ''skewness'' (γ_1''') of the dispersion curve γ_k at the vicinity of the point $k=1$ and on its slope far inside the stability region (γ_2'). Besides, to arrive at the specified accuracy $O(\varepsilon^3)$ in the dispersion equation (21) the quantity γ_k' in the first term on the right-hand side of Eq. (21d) must be expanded in powers of κ to order ε inclusively. Let us remember now that at small κ we have $\gamma_k'=\gamma_1'+\gamma_1''\kappa+O(\kappa^2)$, where $\gamma_1'=O(\varepsilon)$ and $\gamma_1''=O(1)$, see the discussion of Eq. (23) in the preceding section. Thus, at $\kappa=O(\varepsilon)$ the two first terms in the expansion of γ_k' in powers of κ both are of order ε. The latter means that neither γ_1' nor $\gamma_1''\kappa$ may be neglected in the expansion. In other words the shift of the wave number, maximizing γ_k at finite ε, with respect to the point $k=1$ also yields a contribution to the stability conditions in the leading approximation. All these peculiarities make the pattern stability problem essentially nonlocal and sensitive to very subtle details of the spectrum γ_k.

As it was already mentioned in Sec. II, the actual grounds for the ε-scale-mixing are the presence of terms of order $\varepsilon^{5/2}$ in the evaluated form of determinant (20). In turn, there are only two elements of the determinant of order $\sqrt{\varepsilon}$, namely $\pm 2k(\pm k+p)U_{\pm k}$, while the rest are $O(\varepsilon)$. Being entirely responsible for the ε-scale-mixing, these two elements both stand on the second row of the determinant. Taking into account, finally, that this row is originated in the projection of the evolution equation for perturbation (18) on the slowly varying mode with the wave number equal to p, and that this mode is the additional independent degree of freedom associated with the symmetry transformation (7), we arrive at the conclusion that the additional symmetry is the only cause of the mixing [20].

Let us discuss now possible generalizations of the problem. With this end in view it is convenient to employ the following representation of the governing equation:

$$\frac{\partial U_k}{\partial t}=\gamma_k U_k+\int \alpha_{kk_1k_2}U_{k_1}U_{k_2}\delta_{k-k_1-k_2}dk_1dk_2+\cdots, \tag{28}$$

where presently $U_k(t)$ is the Fourier transform of $u(x,t)$ and δ stands for the δ function. Equations of such a type are well known in the pattern formation analysis, see, e.g., Refs. [4,21]. Equations (1) and (3) are particular cases of Eq. (28) with $\alpha_{kk_1k_2}=-i(k_1+k_2)/2$ and $\alpha_{kk_1k_2}=k_1k_2$, respectively. However, the convenience of Eq. (28) is associated with the fact that steady solutions of this equation may be obtained and their stability may be analyzed without concrete definition of the explicit form of the coefficient $\alpha_{kk_1k_2}$, provided this coefficient is a *smooth* function of k,k_1,k_2. Really, looking for a spatially periodic solution to Eq. (28) that in k space is described by a series of the δ functions, it is easy to integrate the equation. The integration yields a set of equations for U's of the same type as that of Eq. (11). Then, linearizing Eq. (28) about the steady periodic solution, representing perturbations as a Fourier transform of Eq. (18), taking into account that the integral in Eq. (28) is dominated by small neighborhoods of points $k=0,\ \pm 1,\ \pm 2,\ldots,$ and expanding $\alpha_{kk_1k_2}$ in powers of deviations of its arguments from the dominant points, we arrive at the system of linear equations for V's similar to Eq. (19). The only difference between these two systems is that now coefficients describing U-V coupling have a more general form obtained from the above-mentioned expansion of $\alpha_{kk_1k_2}$. The same reasons that forced us to consider first corrections with relative smallness to order $\sqrt{\varepsilon}$ in expansions of terms related to γ_k are valid now for the expansion of $\alpha_{kk_1k_2}$ too. Finally we obtain that besides the sensitivity to subtle details of the spectrum γ_k, the generalized problem (28) is also sensitive to fine characteristics of nonlinear mode coupling. The conclusion is identical to those drawn in cases of the free-slip convection [3,4] and reaction-diffusion systems subjected mean field effects [7]. Extension of Eq. (28) to oscillatory short-wave instability and/or a complex order parameter also is a straightforward matter.

Let us discuss now possible application of the obtained results to two more problems related to liquid crystals [9–14] that were already mentioned in the Introduction. In case of electroconvection in a homeotropically aligned nematic layer (axes of molecules are perpendicular to boundary surfaces) with negative dielectric anisotropy, the electric field, applied across the layer, tries to turn the molecules parallel to the layer's plane, i.e., it conflicts with the orientation imposed by the boundary surfaces. As a result the *Fréedericksz* transition [22] occurs beyond a certain critical value of the electric field, and the equilibrium orientation of the molecules in the midplane and its vicinity becomes tilted. Since there is not any singled out direction in plane of the layer, the system beyond the threshold of the Fréedericksz transition is degenerate with respect to rotations around an axis perpendicular to this plane. Usually the threshold of the Fréedericksz transition lies below the one of the electroconvection, so that close to onset of the convection the quiescent (convectionless) state possesses the desirable additional symmetry, originated in the above-mentioned degeneracy.

Certainly, electroconvection patterns in this system are two-dimensional, contrary to the one-dimensional problem analyzed in the present paper. However, the detailed comparison of the stability spectra of roll-patterns in two- and one-dimensional systems with additional symmetry developed in Ref. [18] indicates identity in all qualitative features of the spectra, provided transformations of the additional groups of symmetry in these systems are parametrized by one continuous scalar quantity [23]. Thus, despite the difference in the spatial dimensionality, electroconvection under the specified conditions may be a good tool to obtain experimental evidence of the discussed peculiarities of Eq. (3), including possible instability of all spatially periodic patterns. Indeed, the very first experimental studies of the phenomena [10–12] detected already spatiotemporal chaos very similar to that observed in computer simulations of pattern dynamics governed by Eq. (3) [12,17,18]. The chaos definitely was originated in the degeneracy caused by the Fréedericksz transition: When the degeneracy was lifted by a magnetic field applied in the layer's plane (the field breaks the rotational symmetry) the chaotic patterns evolved to steady spatially periodic rolls. However the ordering was reversible — as soon as the magnetic field was switched off the chaos was restored [11,12]. The framework of the present paper does not allow us to pay more attention to this question. The detailed discussion of this problem will be the subject of a separate publication.

In case of the permeation of cholesteric or smectic liquid crystals at their motion through a capillary, structures associated with liquid crystal ordering (helical in cholesteric and layered in smectic) are fixed due to anchoring effects at capillary's sidewalls. Thus, the hydrodynamic flow occurs as motion of molecules through a fixed structure [13,14].

Similarity of the problem to the pattern formation in systems with additional symmetry is clearly seen if one associates the liquid crystal ordering with short-wave instability of a disordered state and the hydrodynamic flow with mean field effects [24]. In this case the results discussed above provide us with grounds to expect that coupling of short-wave (liquid crystal ordering) and long-wave (hydrodynamic) fluctuations may destabilize the liquid crystal ordering dramatically and even suppress phase transition to the ordered state at a certain range of values of thermodynamic variables.

Ending the general discussion, we would like to emphasize that an additional group of continuous symmetry changes the pattern formation problem qualitatively. Among other things, it may give rise to scale mixing in perturbative expansions, so that a lowest approximation to initial underlying equations becomes irrelevant and yields a wrong dispersion relation in the corresponding pattern stability problem — the circumstance one should always keep in mind, studying such systems.

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[23] In more complicated cases the full continuous group of symmetry is the direct product of a number of subgroups of lower symmetry: $G=G_1\times G_2\times\cdots\times G_n$. Transformations of each of the subgroup generate Goldstone modes in the relevant pattern stability problem. The number of the Goldstone modes, associated with every subgroup, coincides with the number of scalar quantities, parametrizing the transformations of this subgroup. Similarity in pattern stability spectra of different systems may be expected *only* if the subgroups, entering into the direct products in these systems, are in one-to-one correspon-

dence, so that each of the subgroups connected by the correspondence is parametrized by the same number of the scalar quantities. For example, in the case of two-dimensional square cellular patterns without any additional symmetry the group of spatial translations generates *two* Goldstone modes — one associated with translations along the x axis and another with those along the y axis. Thus, the *number* of Goldstone modes coincides with that for the corresponding problem with additional symmetry governed by Eq. (3), but the specified one-to-one correspondence between the groups of symmetry in the two problems does not hold. For this reason, despite the mentioned coincidence of the number of Goldstone modes, the dispersion equation for the sideband perturbations to the cellular patterns has a form entirely different from that of Eq. (22), viz., $\cdots+(b_{1x}p_x^2+b_{1y}p_y^2)\sigma+b_0p_x^2p_y^2=0$ [21], where the ellipsis denotes terms higher order in σ; p_x and p_y stand, respectively, for x and y components of the perturbation's wave vector $\mathbf{p}$ and coefficients b_{1x},b_{1y},b_0 do not depend on $\mathbf{p}$. It is clear that properties of σ defined by this equation have nothing in common with those described by Eq. (23).

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